

# Seymour's second neighbourhood conjecture for quasi-transitive oriented graphs\*

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## Abstract

Seymour's second neighbourhood conjecture asserts that every oriented graph has a vertex whose second out-neighbourhood is at least as large as its out-neighbourhood. In this paper, we prove that the conjecture holds for quasi-transitive oriented graphs, which is a superclass of tournaments and transitive acyclic digraphs. A digraph  $D$  is called quasi-transitive if for every pair  $xy, yz$  of arcs between distinct vertices  $x, y, z$ ,  $xz$  or  $zx$  ("or" is inclusive here) is in  $D$ .

## 1 Introduction

For convenience of the reader we provide all necessary terminology and notation in one section, Section 2.

One of the most interesting and challenging open questions concerning digraphs is Seymour's Second Neighbourhood Conjecture (SSNC) [5], which asserts that one can always find, in an oriented graph  $D$ , a vertex  $x$  whose second out-neighbourhood is at least as large as its out-neighbourhood, i.e.  $|N^+(x)| \leq |N^{++}(x)|$ . Following [4], we will call such a vertex  $x$  a *Seymour vertex*.

Observe that SSNC is not true for digraphs in general. Consider  $\overleftrightarrow{K}_n$ , the complete digraph on  $n$  vertices. For each vertex  $v \in V(\overleftrightarrow{K}_n)$ ,  $N_{\overleftrightarrow{K}_n}^+(v) = V(\overleftrightarrow{K}_n) \setminus \{v\}$  while  $N_{\overleftrightarrow{K}_n}^{++}(v) = \emptyset$ . The conjecture trivially holds for digraphs

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$D$  which contain a vertex of out-degree zero, e.g. for acyclic digraphs. Indeed,  $N_D^+(v_n) = N_D^{++}(v_n) = \emptyset$ .

The first non-trivial result for SSNC was obtained by Fisher [7] who proved Dean's conjecture [5], which is SSNC restricted to tournaments. Fisher used Farkas' Lemma and averaging arguments.

**Theorem 1.1.** [7] *In any tournament  $T$ , there is a vertex  $v$  such that  $|N_T^+(v)| \leq |N_T^{++}(v)|$ .*

A more elementary proof of SSNC for tournaments was given by Havet and Thomassé [9] who introduced a median order approach. Their proof also yields the following stronger result.

**Theorem 1.2.** [9] *A tournament  $T$  with no vertex of out-degree zero has at least two vertices  $v$  such that  $|N_T^+(v)| \leq |N_T^{++}(v)|$ .*

Fidler and Yuster [6] further developed the median order approach and proved that SSNC holds for oriented graphs  $D$  with minimum degree  $|V(D)| - 2$ , tournaments minus a star, and tournaments minus the arc set of a subtournament. The median order approach was also used by Ghazal [8] who proved a weighted version of SSNC for tournaments missing a generalized star. Kaneko and Locke [10] proved SSNC for oriented graphs with minimum out-degree at most 6. Cohn, Godbole, Wright Harkness, and Zhang [4] proved that the conjecture holds for random oriented graphs.

Another approach to SSNC is to determine the maximum value  $\gamma$  such that in every oriented graph  $D$ , there exists a vertex  $x$  such that  $|N_D^+(x)| \leq \gamma |N_D^{++}(x)|$ . SSNC asserts that  $\gamma = 1$ . Chen, Shen, and Yuster [3] proved that  $\gamma \geq r$  where  $r = 0.657298\dots$  is the unique real root of  $2x^3 + x^2 - 1 = 0$ . They also claim a slight improvement to  $r \geq 0.67815\dots$

In this paper, we consider Seymour's Second Neighbourhood Conjecture for quasi-transitive digraphs. This class of digraphs was extensively studied in the literature, see, e.g., a forthcoming book chapter [11]. To obtain our results, we use the median order approach as well as a decomposition theorem of Bang-Jensen and Huang [2] for quasi-transitive digraphs, Theorem 2.1. We also use some structural properties of extended tournaments, a subclass of quasi-transitive oriented graphs, e.g. we refined the median order notion for extended tournaments in Lemma 3.3.

## 2 Terminology and Notation

We will assume that the reader is familiar with the standard terminology on digraphs and refer to [1] for terminology not discussed here. In this paper, all digraphs have no multiple arcs or loops.

We denote the vertex set and the arc set of a digraph  $D$  by  $V(D)$  and  $A(D)$ , respectively. For a vertex subset  $X$ , we denote by  $D\langle X \rangle$  the subdigraph of  $D$  induced by  $X$ ,  $D\langle V(D) - X \rangle$  by  $D - X$ . In addition,  $D - x = D - \{x\}$  for a vertex  $x$  of  $D$ .

Let  $x, y$  be distinct vertices in  $D$ . If there is an arc from  $x$  to  $y$  then we say that  $x$  *dominates*  $y$ , write  $x \rightarrow y$  and call  $y$  (respectively,  $x$ ) an *out-neighbour* (respectively, an *in-neighbour*) of  $x$  (respectively,  $y$ ). For a subdigraph or simply a vertex subset  $H$  of  $D$  (possibly,  $H = D$ ), we let  $N_H^+(x)$  (respectively,  $N_H^-(x)$ ) denote the set of out-neighbours (respectively, the set of in-neighbours) of  $x$  in  $H$  and call it *out-neighbourhood* (respectively, *in-neighbourhood*) of  $x$  in  $H$ . Furthermore,  $d_H^+(x) = |N_H^+(x)|$  (respectively,  $d_H^-(x) = |N_H^-(x)|$ ) is called the *out-degree* (respectively, *in-degree*) of  $x$ . Let

$$N_H^{++}(x) = \bigcup_{u \in N_H^+(x)} N_H^+(u) \setminus N_H^+(x),$$

which is called the *second out-neighbourhood* of  $x$  in  $H$ .

A digraph  $D$  is said to be *strong*, if for every pair of vertices  $x$  and  $y$ ,  $D$  contains a directed path from  $x$  to  $y$  and a directed path from  $y$  to  $x$ . A strong component of a digraph  $D$  is a maximal induced subdigraph of  $D$  which is strong. If  $D_1, \dots, D_t$  are the strong components of  $D$ , then clearly  $V(D_1) \cup \dots \cup V(D_t) = V(D)$  (a digraph with only one vertex is strong). Moreover, we must have  $V(D_i) \cap V(D_j) = \emptyset$  for every  $i \neq j$ . The strong components of  $D$  can be labelled  $D_1, \dots, D_t$  such that there is no arc from  $D_j$  to  $D_i$  unless  $j < i$ . We call such an ordering an *acyclic ordering of the strong components* of  $D$ .

A digraph  $D$  is *acyclic* if it has no directed cycle. An ordering  $v_1, v_2, \dots, v_n$  of vertices of a digraph  $D$  is called *acyclic* if for every arc  $v_i v_j \in A(D)$ , we have  $i < j$ . It is well-known that every acyclic digraph has an acyclic ordering [1]. Clearly, an acyclic ordering is a median order for acyclic digraphs.

Let  $D$  be a digraph with vertex set  $\{v_1, v_2, \dots, v_n\}$ , and let  $G_1, G_2, \dots, G_n$  be digraphs which are pairwise vertex disjoint. The *composition*  $D[G_1, G_2, \dots, G_n]$  is the digraph  $L$  with vertex set  $V(G_1) \cup V(G_2) \cup \dots \cup V(G_n)$  and arc set  $(\bigcup_{i=1}^n A(G_i)) \cup \{g_i g_j \mid g_i \in V(G_i), g_j \in V(G_j), v_i v_j \in A(D)\}$ . If  $D = H[S_1, S_2, \dots, S_h]$  and none of the digraphs  $S_1, S_2, \dots, S_h$  has an arc, then  $D$  is an *extension* of  $H$ . For  $i \in \{1, 2, \dots, s\}$ , each  $S_i$  called the *partite set* of  $D$ .

An *oriented graph* is a digraph with no cycle of length two. A *tournament* is an oriented graph where every pair of distinct vertices are adjacent. An *extended tournament* is an extension of a tournament.

A digraph  $D$  is *quasi-transitive* if for every pair  $xy$  and  $yz$  of arcs in  $D$  with  $x \neq z$  implies that  $x$  and  $y$  are adjacent. A digraph  $D$  is *transitive* if, for every pair  $xy$  and  $yz$  of arcs in  $D$  with  $x \neq z$ , the arc  $xz$  is also in  $D$ . Observe that each transitive digraph is quasi-transitive and each extended tournament is also quasi-transitive.

To make quasi-transitive digraphs easier to deal with, Bang-Jensen and Huang [2] introduced the following characterization of this class of digraphs.

**Theorem 2.1.** [2] *Let  $D$  be a quasi-transitive digraph.*

- *If  $D$  is not strong, then there exists a transitive oriented graph  $T$  with vertices  $\{u_1, u_2, \dots, u_t\}$  and strong quasi-transitive digraphs  $H_1, H_2, \dots, H_t$  such that  $D = T[H_1, H_2, \dots, H_t]$ , where  $H_i$  is substituted for  $u_i, i \in \{1, 2, \dots, t\}$ .*

- If  $D$  is strong, then there exists a strong semicomplete digraph  $S$  with vertices  $\{v_1, v_2, \dots, v_s\}$  and quasi-transitive digraphs  $Q_1, Q_2, \dots, Q_s$  such that  $Q_i$  is either a vertex or is non-strong and  $D = S[Q_1, Q_2, \dots, Q_s]$ , where  $Q_i$  is substituted for  $v_i, i \in \{1, 2, \dots, s\}$ .

The decomposition described in Theorem 2.1 is called the *canonical decomposition* of the quasi-transitive digraph  $D$ .

### 3 Preliminary results

In this paper, we use median orders to prove the main results. A *median order* of a digraph  $D$  is a linear order  $(v_1, v_2, \dots, v_n)$  of its vertex set such that  $|\{(v_i, v_j) : i < j\}|$  (the number of arcs directed from left to right) is as large as possible. It is a very useful tool for proving results about tournaments and other classes of digraphs. The following two basic properties of median orders of tournaments are given by Havet and Thomassé [9], which are used to prove Theorems 1.1 and 1.2.

**Lemma 3.1.** [9] *Let  $T$  be a tournament and  $(v_1, v_2, \dots, v_n)$  a median order of  $T$ . Then, for any two indices  $i, j$  with  $1 \leq i < j \leq n$ :*

- (M1) *the suborder  $(v_i, v_{i+1}, \dots, v_j)$  is a median order of the induced subtournament  $T[v_i, v_{i+1}, \dots, v_j]$ ;*
- (M2) *vertex  $v_i$  dominates at least half of the vertices  $v_{i+1}, v_{i+2}, \dots, v_j$ , and vertex  $v_j$  is dominated by at least half of the vertices  $v_i, v_{i+1}, \dots, v_{j-1}$ .*

Now we give the following easy but useful observation, which indicates the relationship between the Seymour vertex of a quasi-transitive oriented graph and the one of an extended tournament.

**Lemma 3.2.** *Let  $D$  be a strong quasi-transitive oriented graph and  $D = S[Q_1, Q_2, \dots, Q_s]$  be the canonical decomposition. Let  $D^* = S[V_1, V_2, \dots, V_s]$  be an extended tournament, where  $V_i$  is the vertex set of the subdigraph  $Q_i$  for  $i \in \{1, 2, \dots, s\}$ . If there is a vertex  $x \in V_i$  such that  $x$  is a Seymour vertex of  $Q_i$  and a Seymour vertex of  $D^*$ , then  $x$  is a Seymour vertex of  $D$ .*

*Proof.* Since  $x$  is a Seymour vertex in  $Q_i$  and a Seymour vertex in  $D^*$ , we have

$$|N_{Q_i}^+(x)| \leq |N_{Q_i}^{++}(x)|, \quad |N_{D^*}^+(x)| \leq |N_{D^*}^{++}(x)|.$$

Clearly,

$$N_D^+(x) = N_{Q_i}^+(x) \cup N_{D^*}^+(x), \quad N_D^{++}(x) = N_{Q_i}^{++}(x) \cup N_{D^*}^{++}(x)$$

Thus  $|N_D^+(x)| \leq |N_D^{++}(x)|$ . □

Observe that all vertices in a partite set of an extended tournament have the same adjacency to the vertices in another partite set. A median order in an extended tournament can be chosen with the following property.

**Lemma 3.3.** *Let  $D = S[V_1, V_2, \dots, V_s]$  be an extended tournament. Then there is a median order  $\sigma = (v_1, v_2, \dots, v_n)$  of  $D$  such that for all  $i \in \{1, 2, \dots, s\}$ , the vertices in  $V_i$  are a continuous segment in the median order.*

*Proof.* Suppose that  $v_\alpha, v_\beta$  are two vertices in some  $V_i$  with  $\beta - \alpha \geq 2$ . Let  $W = \{v_{\alpha+1}, \dots, v_{\beta-1}\}$ . We will show that  $|N_W^+(v_\beta)| = |N_W^-(v_\beta)|$ . Clearly,  $N_W^+(v_\beta) = N_W^+(v_\alpha)$  and  $N_W^-(v_\beta) = N_W^-(v_\alpha)$ .

If  $|N_W^+(v_\beta)| > |N_W^-(v_\beta)|$ , then

$$(v_1, \dots, v_\alpha, v_\beta, v_{\alpha+1}, \dots, v_{\beta-1}, v_{\beta+1}, \dots, v_n)$$

is a linear order such that  $|\{(v_i, v_j) : i < j\}|$  is larger than  $\sigma$ , a contraction.

If  $|N_W^+(v_\beta)| < |N_W^-(v_\beta)|$ , then  $|N_W^+(v_\alpha)| < |N_W^-(v_\alpha)|$ . Now

$$(v_1, \dots, v_{\alpha-1}, v_{\alpha+1}, \dots, v_{\beta-1}, v_\alpha, v_\beta, v_{\beta+1}, \dots, v_n)$$

is a linear order such that  $|\{(v_i, v_j) : i < j\}|$  is larger than  $\sigma$ , a contraction.

So  $|N_W^+(v_\beta)| = |N_W^-(v_\beta)|$  and  $(v_1, \dots, v_\alpha, v_\beta, v_{\alpha+1}, \dots, v_{\beta-1}, v_{\beta+1}, \dots, v_n)$  is also a median order of  $D$ . By induction on  $|V_i|$ , the lemma follows.  $\square$

For an extended tournament  $D$ , we call a median order *well-organized* if it satisfies the condition of Lemma 3.3. Similarly to Lemma 3.1, we have two basic properties of median orders of extended tournaments.

**Lemma 3.4.** *Let  $D = S[V_1, V_2, \dots, V_s]$  be an extended tournament and  $\sigma = (v_1, v_2, \dots, v_n)$  be a well-organized median order of  $D$ . Then, for any two indices  $i, j$  with  $1 \leq i < j \leq n$ :*

(W1) *the suborder  $(v_i, v_{i+1}, \dots, v_j)$  is a well-organized median order of the induced extended tournament  $D\langle v_i, v_{i+1}, \dots, v_j \rangle$ ;*

(W2)  *$|N_U^+(v_i)| \geq |N_U^-(v_i)|$  and  $|N_U^+(v_j)| \leq |N_U^-(v_j)|$ , where  $U = \{v_i, v_{i+1}, v_{i+2}, \dots, v_j\}$ .*

*Proof.* (W1) Suppose that  $(v_i, v_{i+1}, \dots, v_j)$  is not a median order of the induced extended tournament  $T\langle v_i, v_{i+1}, \dots, v_j \rangle$ . Let  $(u_i, u_{i+1}, \dots, u_j)$  be a median order of the subdigraph. So it has more arcs directed from left to right. Now

$$\tilde{\sigma} = (v_1, \dots, v_{i-1}, u_i, \dots, u_j, v_{j+1}, \dots, v_n)$$

is a linear order of  $D$  which has more arcs directed from left to right than  $\sigma$ . By the proof of Lemma 3.3, we can get a well-organized median order from  $\tilde{\sigma}$ , a contraction. So  $(v_i, v_{i+1}, \dots, v_j)$  is a median order of  $T\langle v_i, v_{i+1}, \dots, v_j \rangle$ . Clearly,  $(v_i, v_{i+1}, \dots, v_j)$  is well-organized. Therefore, (W1) holds.

(W2) Suppose that  $|N_W^+(v_i)| < |N_W^-(v_i)|$ . Then the linear order

$$\bar{\sigma} = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_j, v_i, v_{j+1}, \dots, v_n)$$

has more arcs directed from left to right than  $\sigma$ , a contraction. So  $|N_W^+(v_i)| \geq |N_W^-(v_i)|$ . Similarly,  $|N_U^+(v_j)| \leq |N_U^-(v_j)|$  holds.  $\square$

## 4 Main Results

First we deal with SSNC for extended tournaments. Let  $D = S[V_1, V_2, \dots, V_s]$  be an extended tournament and  $\sigma = (v_1, v_2, \dots, v_n)$  be a well-organized median order of  $D$ . Let us distinguish two types of vertices of  $N_D^-(v_n)$ : a vertex  $v_j \in N_D^-(v_n)$  is  $\sigma$ -good if there exists  $v_i \in N_D^+(v_n)$ , with  $i < j$ , such that  $v_i \rightarrow v_j$ ; otherwise  $v_j$  is  $\sigma$ -bad. Let  $G_\sigma$  be the set of  $\sigma$ -good vertices. This excellent idea comes from Havet and Thomassé in [9]. The following theorem shows that we can generalize Theorem 1.1 to extended tournaments.

**Theorem 4.1.** *Let  $D = S[V_1, V_2, \dots, V_s]$  be an extended tournament and  $\sigma = (v_1, v_2, \dots, v_n)$  be a well-organized median order of  $D$ . Then  $|N_D^+(v_n)| \leq |G_\sigma| \leq |N_D^{++}(v_n)|$ .*

*Proof.* Observe that  $G_\sigma \subseteq N_D^{++}(v_n)$  and it is enough to prove that  $|N_D^+(v_n)| \leq |G_\sigma|$ . We will prove it by induction on  $n$ . The case  $n = 1$  holds vacuously. Assume now  $n > 1$ .

If there is no  $\sigma$ -bad vertex, then  $G_\sigma = N_D^-(v_n)$ . Moreover, by the property (W2) of Lemma 3.4,  $|N_D^+(v_n)| \leq |N_D^-(v_n)|$ , so the conclusion holds. Assume now that there exists a  $\sigma$ -bad vertex. Let  $i$  be the smallest integer such that  $v_i$  is  $\sigma$ -bad and  $j$  be the largest integer such that  $v_j$  is in the same partite set as  $v_i$ . Observe that each vertex of  $\{v_i, \dots, v_j\}$  is  $\sigma$ -bad. Set  $D_r = D[\{v_{j+1}, \dots, v_n\}]$ . By (W1) of Lemma 3.4,  $\sigma_r = (v_{j+1}, \dots, v_n)$  is a well-organized median order of  $D_r$ . By the induction hypothesis,  $|N_{D_r}^+(v_n)| \leq |G_{\sigma_r}|$ . Since every  $\sigma_r$ -good vertex is also  $\sigma$ -good, we get

$$|N_D^+(v_n) \cap \{v_{j+1}, \dots, v_n\}| \leq |G_\sigma \cap \{v_{j+1}, \dots, v_n\}|. \quad (1)$$

By the minimality of the index of  $i$ , every vertex of  $\{v_1, \dots, v_{i-1}\}$  is either in  $G_\sigma$  or in  $N_D^+(v_n)$ . Moreover, since  $v_i$  is  $\sigma$ -bad, we have  $N_D^+(v_n) \cap \{v_1, \dots, v_{i-1}\} \subseteq N_D^+(v_i) \cap \{v_1, \dots, v_{i-1}\}$ , so  $G_\sigma \cap \{v_1, \dots, v_{i-1}\} \supseteq N_D^-(v_i) \cap \{v_1, \dots, v_{i-1}\}$ . Now by (W2) of Lemma 3.4,  $|N_D^-(v_i) \cap \{v_1, \dots, v_{i-1}\}| \geq |N_D^+(v_i) \cap \{v_1, \dots, v_{i-1}\}|$ . Hence

$$|N_D^+(v_n) \cap \{v_1, \dots, v_{i-1}\}| \leq |N_D^-(v_i) \cap \{v_1, \dots, v_{i-1}\}| \leq |G_\sigma \cap \{v_1, \dots, v_{i-1}\}|.$$

Clearly,  $N_D^+(v_n) \cap \{v_i, \dots, v_j\} = \emptyset$  and  $G_\sigma \cap \{v_i, \dots, v_j\} = \emptyset$ . So

$$|N_D^+(v_n) \cap \{v_1, \dots, v_j\}| \leq |G_\sigma \cap \{v_1, \dots, v_j\}|. \quad (2)$$

Thus  $|N_D^+(v_n)| \leq |G_\sigma|$ .  $\square$

The following theorem shows that we can generalize Theorem 1.1 to quasi-transitive oriented graphs.

**Theorem 4.2.** *In any quasi-transitive oriented graph  $D$ , there is a vertex  $v$  such that  $|N_D^+(v)| \leq |N_D^{++}(v)|$ .*

*Proof.* The proof is by induction on the order  $n$  of  $D$ . It is easy to check that the cases  $1 \leq n \leq 3$  hold. Assume that  $n \geq 4$ .

*Case 1:*  $D$  is not strong. Let  $D = T[H_1, H_2, \dots, H_t]$  be the canonical decomposition of  $D$ , where  $T$  is a transitive oriented graph and  $H_i$  is a strong quasi-transitive oriented graph for  $i \in \{1, 2, \dots, t\}$ . Without loss of generality, assume that  $H_1, H_2, \dots, H_t$  is the acyclic ordering of the strong components of  $D$ . By induction hypothesis, let  $v$  be a Seymour vertex of  $H_t$ . This means  $|N_{H_t}^+(v)| \leq |N_{H_t}^{++}(v)|$ . Clearly,  $N_D^+(v) = N_{H_t}^+(v)$  and  $N_D^{++}(v) = N_{H_t}^{++}(v)$ . Thus  $|N_D^+(v)| \leq |N_D^{++}(v)|$ .

*Case 2:*  $D$  is strong. Let  $D = S[Q_1, Q_2, \dots, Q_s]$  be the canonical decomposition of  $D$ , where  $S$  is a strong tournament and  $Q_i$  is a single vertex or non-strong quasi-transitive oriented graph for  $i \in \{1, 2, \dots, s\}$ . Let  $D^* = S[V_1, V_2, \dots, V_s]$  be an extended tournament, where  $V_i$  is the vertex set of the subdigraph  $Q_i$  for  $i \in \{1, 2, \dots, s\}$ . By Theorem 4.1,  $D^*$  has a Seymour vertex  $v$ . Assume  $v \in V_i$ . Then each vertex in  $V_i$  is a Seymour vertex in  $D^*$ . By induction hypothesis, there is a Seymour vertex of  $Q_i$ , say also  $v$ . By Lemma 3.2,  $v$  is a Seymour vertex in  $D$  and  $|N_D^+(v)| \leq |N_D^{++}(v)|$ .  $\square$

To generalize Theorem 1.2 to extended tournaments, we need the notion of sedimentation of a median order, which is introduced in [9]. Let  $D = S[V_1, V_2, \dots, V_s]$  be an extended tournament and  $\sigma = (v_1, v_2, \dots, v_n)$  be a well-organized median order of  $D$ . Let  $G_\sigma$  be the set of  $\sigma$ -good vertices. Recall that by Theorem 4.1,  $|N_D^+(v_n)| \leq |G_\sigma|$ . A *sedimentation* of  $\sigma$ , denoted by  $Sed(\sigma)$ , is a linear order satisfying:

- If  $|N_D^+(v_n)| < |G_\sigma|$ , then  $Sed(\sigma) = \sigma$ ;
- If  $|N_D^+(v_n)| = |G_\sigma|$ , we denote by  $b_1, \dots, b_k$  the  $\sigma$ -bad vertices, by  $v_l, \dots, v_n$  the vertices of the partite set containing  $v_n$  and by  $w_1, \dots, w_{l-k-1}$  the vertices of  $N_D^+(v_n) \cup G_\sigma$ , enumerated in increasing order with respect to  $\sigma$ . In this case,  $Sed(\sigma)$  is the order  $(b_1, \dots, b_k, v_l, \dots, v_n, w_1, \dots, w_{l-k-1})$ .

**Lemma 4.3.** *If  $\sigma$  is a well-organized median order of an extended tournament  $D$ , then  $Sed(\sigma)$  is also a well-organized median order of  $D$ .*

*Proof.* Let  $D = S[V_1, V_2, \dots, V_s]$  be an extended tournament and  $\sigma = (v_1, v_2, \dots, v_n)$  be a well-organized median order of  $D$ . If  $Sed(\sigma) = \sigma$ , there is nothing to prove. So we assume that  $|N_D^+(v_n)| = |G_\sigma|$ .

The proof is by induction on the number  $k$  of  $\sigma$ -bad vertices. If  $k = 0$ , all the vertices in  $V(D) \setminus \{v_l, \dots, v_n\}$  are  $\sigma$ -good or in  $N_D^+(v_n)$ , in particular  $N_D^-(v_n) = G_\sigma$ . So  $|N_D^+(v_n)| = |G_\sigma| = |N_D^-(v_n)|$  and the equality holds for all vertices in  $\{v_l, \dots, v_n\}$ . Thus, the order  $Sed(\sigma) = (v_l, \dots, v_n, v_1, \dots, v_{l-1})$  is a median order of  $D$ . Clearly, it is well-organized.

Assume now that  $k$  is a positive integer. Let  $i$  be the smallest index of a  $\sigma$ -bad vertex and  $j$  be the largest integer such that  $v_j$  is in the same partite set as  $v_i$ . We will prove that  $|N_D^+(v_i) \cap \{v_1, \dots, v_{i-1}\}| = |N_D^-(v_i) \cap \{v_1, \dots, v_{i-1}\}|$ . By (W2) of Lemma 3.4,

$$|N_D^+(v_i) \cap \{v_1, \dots, v_{i-1}\}| \leq |N_D^-(v_i) \cap \{v_1, \dots, v_{i-1}\}|. \quad (3)$$

By (1) and (2),  $|N_D^+(v_n) \cap \{v_{j+1}, \dots, v_n\}| \leq |G_\sigma \cap \{v_{j+1}, \dots, v_n\}|$  and  $|N_D^+(v_n) \cap \{v_1, \dots, v_j\}| \leq |G_\sigma \cap \{v_1, \dots, v_j\}|$ . By assumption,  $|N_D^+(v_n)| = |G_\sigma|$ , and then

$$\begin{aligned} & |N_D^+(v_n) \cap \{v_1, \dots, v_j\}| + |N_D^+(v_n) \cap \{v_{j+1}, \dots, v_n\}| \\ &= |G_\sigma \cap \{v_1, \dots, v_j\}| + |G_\sigma \cap \{v_{j+1}, \dots, v_n\}|. \end{aligned}$$

Hence  $|N_D^+(v_n) \cap \{v_{j+1}, \dots, v_n\}| = |G_\sigma \cap \{v_{j+1}, \dots, v_n\}|$  and  $|N_D^+(v_n) \cap \{v_1, \dots, v_j\}| = |G_\sigma \cap \{v_1, \dots, v_j\}|$ . Note that  $N_D^+(v_n) \cap \{v_i, \dots, v_j\} = \emptyset$  and  $G_\sigma \cap \{v_i, \dots, v_j\} = \emptyset$ . So

$$|N_D^+(v_n) \cap \{v_1, \dots, v_{i-1}\}| = |G_\sigma \cap \{v_1, \dots, v_{i-1}\}|.$$

Since  $v_i$  is  $\sigma$ -bad,  $N_D^+(v_n) \cap \{v_1, \dots, v_{i-1}\} \subseteq N_D^+(v_i) \cap \{v_1, \dots, v_{i-1}\}$  and so  $N_D^-(v_n) \cap \{v_1, \dots, v_{i-1}\} \supseteq N_D^-(v_i) \cap \{v_1, \dots, v_{i-1}\}$ . By definition of  $i$ ,  $N_D^-(v_n) \cap \{v_1, \dots, v_{i-1}\} = G_\sigma \cap \{v_1, \dots, v_{i-1}\}$ . Hence,

$$|N_D^-(v_i) \cap \{v_1, \dots, v_{i-1}\}| \leq |G_\sigma \cap \{v_1, \dots, v_{i-1}\}| \leq |N_D^+(v_i) \cap \{v_1, \dots, v_{i-1}\}|.$$

The above inequality and (3) imply  $|N_D^+(v_i) \cap \{v_1, \dots, v_{i-1}\}| = |N_D^-(v_i) \cap \{v_1, \dots, v_{i-1}\}|$ . Clearly, equality holds for each vertex in  $\{v_i, \dots, v_j\}$ . So

$$(v_i, \dots, v_j, v_1, \dots, v_{i-1}, v_{j+1}, \dots, v_n)$$

is a well-organized median order of  $D$ . Applying the induction hypothesis to the median order  $(v_1, \dots, v_{i-1}, v_{j+1}, \dots, v_n)$  which has less bad vertices than  $\sigma$ , we obtain the result.  $\square$

The following theorem indicates that an extended tournament always has two vertices with large second out-neighbourhood, provided that every vertex has out-degree at least 1 and, the second out-neighbourhood of Seymour vertex is more than out-neighbourhood if such two Seymour vertices are in a same partite set.

**Theorem 4.4.** *Let  $D = S[V_1, V_2, \dots, V_s]$  be an extended tournament and let  $\sigma = (v_1, v_2, \dots, v_n)$  be a well-organized median order of  $D$ . If  $D$  has no vertex of out-degree zero, then*

- (a) *there are at least two vertices  $v$  such that  $|N_D^+(v)| \leq |N_D^{++}(v)|$ , and*
- (b)  *$|N_D^+(v_n)| < |N_D^{++}(v_n)|$  unless there is another Seymour vertex  $u$  which is in a distinct partite set from  $v_n$ .*

*Proof.* (a) By Theorem 4.1,  $v_n$  is a Seymour vertex, so we need to find another vertex with this property. If the partite set containing  $v_n$  has at least two vertices, we are done. So assume that  $v_n$  is the unique vertex in it.

Let  $\tilde{D} = D - v_n$ . Note that if  $(u_1, \dots, u_{n-1})$  is a well-organized median order of  $\tilde{D}$ , then  $(u_1, \dots, u_{n-1}, v_n)$  is also a well-organized median order of  $D$ , and always  $u_{n-1} \rightarrow v_n$  in  $D$ .



*Case 1:*  $\tilde{D}$  has a well-organized median order  $\tilde{\sigma} = (u_1, \dots, u_{n-1})$  such that  $\tilde{\sigma} = \text{Sed}(\tilde{\sigma})$ . By the definition of sedimentation,  $|N_{\tilde{D}}^+(u_{n-1})| < |G_{\tilde{\sigma}}|$ . Then

$$|N_D^+(u_{n-1})| = |N_{\tilde{D}}^+(u_{n-1})| + 1 \leq |G_{\tilde{\sigma}}| \leq |N_{\tilde{D}}^{++}(u_{n-1})| \leq |N_D^{++}(u_{n-1})|.$$

*Case 2:* For every median order  $\tilde{\sigma}$  of  $\tilde{D}$ ,  $\tilde{\sigma} \neq \text{Sed}(\tilde{\sigma})$ . Now define inductively  $\sigma_0 = (v_1, \dots, v_{n-1})$  and  $\sigma_{q+1} = \text{Sed}(\sigma_q)$ . By (W1) of Lemma 3.4,  $\sigma_0$  is a well-organized median order of  $\tilde{D}$ ; Lemma 4.3 implies that  $\sigma_q$  is a well-organized median order of  $\tilde{D}$  for every positive integer  $q$ . Since  $D$  has no vertex of out-degree zero,  $v_n$  has an out-neighbour  $v_j$ . Recall that, for each  $q$ , the last vertex of  $\sigma_q$  dominates  $v_n$ . So  $v_j$  is not the last vertex of any  $\sigma_q$ . Observe that there is  $q$  such that  $v_j$  is  $\sigma_q$ -bad, for otherwise the index of  $v_j$  would always increase. Let  $\sigma_q = (u_1, \dots, u_{n-1})$ . Then by Theorem 4.1,

$$|N_D^+(u_{n-1})| = |N_{\tilde{D}}^+(u_{n-1})| + 1 \leq |G_{\sigma_q}| + 1.$$

Moreover  $u_{n-1} \rightarrow v_n \rightarrow v_j$  and then  $v_j \in N_D^{++}(u_{n-1})$ . The second neighbourhood of  $u_{n-1}$  has at least  $|G_{\sigma_q} \cup \{v_j\}| = |G_{\sigma_q}| + 1$  vertices. Thus  $|N_D^+(u_{n-1})| \leq |N_D^{++}(u_{n-1})|$ .

(b) Suppose that  $|N_D^+(v_n)| = |N_D^{++}(v_n)|$ . By the proof of Theorem 4.1, the equality  $|N_D^+(v_n)| = |G_\sigma| = |N_D^{++}(v_n)|$  must hold. So the sedimentation  $\text{Sed}(\sigma)$  of  $\sigma$  is  $(b_1, \dots, b_k, v_l, \dots, v_n, w_1, \dots, w_{l-k-1})$ , where  $b_1, \dots, b_k$  are the  $\sigma$ -bad vertices,  $v_l, \dots, v_n$  are the vertices of the partite set containing  $v_n$  and  $w_1, \dots, w_{l-k-1}$  are the vertices of  $N_D^+(v_n) \cup G_\sigma$ . Lemma 4.3 yields  $\text{Sed}(\sigma)$  is also a well-organized median order. Since  $D$  has no vertex of out-degree zero,  $N_D^+(v_n)$  is not an empty set. Then  $w_{l-k-1}$  exists and is another Seymour vertex which is in a distinct partite set from  $v_n$ .  $\square$

Now we show that a quasi-transitive oriented graph always has two vertices with large second out-neighbourhood, provided that every vertex has out-degree at least 1.

**Theorem 4.5.** *A quasi-transitive oriented graph  $D$  with no vertex of out-degree zero has at least two vertices  $v$  such that  $|N_D^+(v)| \leq |N_D^{++}(v)|$ .*

*Proof.* The proof is by induction on the order  $n$  of  $D$ . It is easy to check that the case  $n = 3$  holds. Assume that  $n \geq 4$ .

*Case 1:*  $D$  is not strong. Let  $D = T[H_1, H_2, \dots, H_t]$  be the canonical decomposition of  $D$ , where  $T$  is a transitive oriented graph and  $H_i$  is a strong quasi-transitive oriented graph for  $i \in \{1, 2, \dots, t\}$ . Without loss of generality, assume that  $H_1, H_2, \dots, H_t$  is the acyclic ordering of the strong components of  $D$ . Since  $D$  has no vertex out-degree zero, the last component  $H_t$  must contain at least three vertices. This means  $H_t$  is a quasi-transitive oriented graph with no vertex of out-degree zero. By induction hypothesis, there are at least two Seymour vertices in  $H_t$ . Since every Seymour vertex of  $H_t$  is also a Seymour vertex of  $D$ ,  $D$  has at least two vertices  $v$  such that  $|N_D^+(v)| \leq |N_D^{++}(v)|$ .

*Case 2:*  $D$  is strong. Let  $D = S[Q_1, Q_2, \dots, Q_s]$  be the canonical decomposition of  $D$ , where  $S$  is a strong tournament and  $Q_i$  is a single vertex or non-strong quasi-transitive oriented graph for  $i \in \{1, 2, \dots, s\}$ . Let  $D^* = S[V_1, V_2, \dots, V_s]$  be an extended tournament, where  $V_i$  is the vertex set of the subdigraph  $Q_i$  for  $i \in \{1, 2, \dots, s\}$ . Clearly,  $D^*$  is strong and hence has no vertex of out-degree zero. Let  $\sigma = (v_1, v_2, \dots, v_n)$  be a well-organized median order of  $D^*$ . By Theorem 4.4(b),  $|N_{D^*}^+(v_n)| < |N_{D^*}^{++}(v_n)|$  unless there is another Seymour vertex  $u$  which is in a distinct partite set from  $v_n$ . For the case when the latter holds,  $D^*$  has two Seymour vertices which belong to different partite sets, say  $V_\alpha$  and  $V_\beta$ . By induction hypothesis, there is a Seymour vertex in each  $Q_i$  for  $i \in \{1, 2, \dots, s\}$ . Now Theorem 3.2 implies that the Seymour vertices of  $Q_\alpha$  and  $Q_\beta$  are also Seymour vertices of  $D$ .

So assume that  $|N_{D^*}^+(v_n)| < |N_{D^*}^{++}(v_n)|$ . For convenience, assume  $v_n \in V_1$ . We claim that the partite set  $V_1$  contains at least two vertices. Indeed, if not, then  $v_n$  is the unique vertex of  $V_1$ . By Theorem 4.4(a), there must exist another Seymour vertex  $u$  which is not in  $V_1$ . As shown above,  $u$  is also a Seymour vertex of  $D$ . So  $V_1$  contains at least two vertices.

If there are at least two Seymour vertices in  $Q_1$ , then they are also Seymour vertices in  $D$ . So assume that  $Q_1$  has exactly one Seymour vertex, say  $v_n$ . Now we claim that there is another vertex  $u \in V_1$  distinct from  $v_n$  such that  $|N_{Q_1}^+(u)| - 1 \leq |N_{Q_1}^{++}(u)|$ . In fact, set  $Q_1 = T_1[Q_1^1, Q_1^2, \dots, Q_1^r]$  be the canonical decomposition of  $Q_1$ , where  $T_1$  is a transitive oriented graph and  $Q_1^i$  is a strong quasi-transitive oriented graph for  $i \in \{1, 2, \dots, r\}$ . Also, assume that  $Q_1^1, Q_1^2, \dots, Q_1^r$  is the acyclic ordering of the strong components of  $Q_1$ . Clearly,  $Q_1^r$  is the unique terminal strong component and  $v_n$  is the unique vertex of  $Q_1^r$ . By induction hypothesis, there is a Seymour vertex  $u$  in  $Q_1^{r-1}$ . This means  $|N_{Q_1^{r-1}}^+(u)| \leq |N_{Q_1^{r-1}}^{++}(u)|$ . Now

$$|N_{Q_1}^+(u)| - 1 = |N_{Q_1^{r-1}}^+(u)| \leq |N_{Q_1^{r-1}}^{++}(u)| \leq |N_{Q_1}^{++}(u)|.$$

Since  $u$  and  $v_n$  are in the same partite set  $V_1$  of  $D^*$ , the inequality  $|N_{D^*}^+(u)| < |N_{D^*}^{++}(u)|$  holds. Clearly,  $N_D^+(u) = N_{Q_1}^+(u) \cup N_{D^*}^+(u)$  and  $N_D^{++}(u) = N_{Q_1}^{++}(u) \cup N_{D^*}^{++}(u)$ . Thus  $|N_D^+(u)| \leq |N_D^{++}(u)|$  and the theorem holds.  $\square$

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